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Ramsey numbers $r(K_3, G)$ for $G \cong K_7 - 2P_2$ and $G \cong K_7 - 3P_2$

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Abstract

For the graphs $K_7 - 2P_2$ and $K_7 - 3P_2$ we give a proof of their triangle-graph Ramsey numbers without using any computer support. © 1998 Published by Elsevier Science B.V. All rights reserved

Keywords: Ramsey numbers; Triangle-free

1. Introduction

We consider finite, undirected graphs $G = (V(G), E(G))$ without loops or multiple edges. By a two-colouring (R, B) we mean a colouring of the edges of G with two different colours, say red and blue. Let $\langle R \rangle$ ($\langle B \rangle$) denote the subgraph of G which is induced by the red (blue) edges. The path, the cycle and the complete graph on p vertices will be denoted by P_p, C_p and K_p, respectively. For short we say that the set X, $X \subseteq V(G)$ spans a red (blue) graph H if the red (blue) edges between the vertices in X contain a subgraph which is isomorphic to H. We define the Ramsey number $r(G_1, G_2)$ of two graphs G_1 and G_2 as the minimum $p \in \mathbb{N}$ such that for each two-colouring of K_p either $\langle R \rangle$ contains a G_1 or $\langle B \rangle$ contains a G_2. In the sequel one of these graphs is always isomorphic to a triangle. We then speak of triangle graph Ramsey numbers.

Already Graver and Yackel (1968) gave a proof of $r(K_3, K_7) = 23$ [3] and Grenda and Harborth (1982) showed that $r(K_3, K_7 - e) = 21$ [4].

In [5] Xia Jin computed the triangle graph Ramsey numbers for all (853) connected graphs on seven vertices. Unfortunately there are some errors in his thesis. In [7] we determined the triangle graph Ramsey number for 812 of these graphs. Meanwhile we obtained a proof of the triangle graph Ramsey numbers for all 853 graphs. It turned

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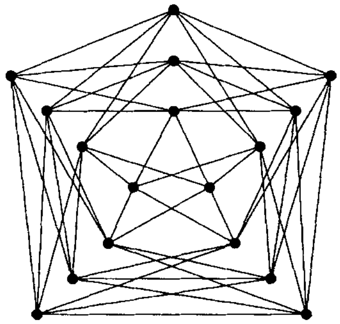
out that the proofs for the cases where $G=K_7-2P_2$ and $G=K_7-3P_3$ are the most difficult ones. In this paper we are going to prove these cases. The complete proofs for the remaining graphs and a list of these graphs will be given in [6].

Using efficient algorithms G. Brinkmann has also recently computed all these Ramsey numbers independently and obtained the same results. In addition he also obtained the triangle graph Ramsey numbers for the connected graphs on eight vertices [1].

Theorem 1. $r(K_3,K_7-2P_2)=r(K_3,K_7-3P_2)=18$.

2. Proof

Like in [4] we will use further notations: For a vertex $v\in V(G)$ we denote its neighbourhood in $\langle R\rangle$ or $\langle B\rangle$ by $N_R(v)$ or $N_B(v)$, respectively. Let $X_R(v)=N_R(v)\cap X$ where X is a subset of $V(G)$ and v is a vertex in $V(G)$ ($X_B(v)=N_B(v)\cap X$ likewise). Xia Jin constructed in [5] the following counterexample on 17 vertices the two-colouring of which contains neither a red triangle nor a blue K_7-3P_2.



Hence we have

- $r(K_3,G)\geqslant 18$ for all connected graphs containing a K_7-3P_2.

It remains to show that

- $r(K_3,G)\leqslant 18$ for all connected graphs $G\subseteq K_7-2P_2$.

Altogether it then follows that

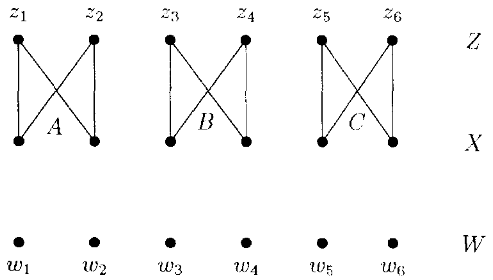
$$r(K_3,K_7-2P_2)=r(K_3,K_7-3P_2)=18.$$

Proof. Suppose that there is a two-colouring of a K_{18} which avoids both a blue K_7-2P_2 and a red triangle. If there is a vertex $v\in V(K_{18})$ with $|N_R(v)|\leqslant 4$ then consider the subgraph H of G where $V(H)=V(K_{18})\setminus(N_R(v)\cup\{v\})$. Since $|V(H)|\geqslant 13$ and $r(K_3,K_6-2P_2)=13$ [2], the blue edges of H span a blue K_6-2P_2. Hence there is a blue K_7-2P_2 in $H\cup\{v\}$. We therefore conclude $|N_R(v)|\geqslant 5$ for all $v\in V(K_{18})$. Now we are going to consider eight different cases:

(1) $|N_R(v)|=5$ for all $v\in V(K_{18})$: $r(K_3,K_6)=18$ implies that there is a blue K_6. Say $X=\{a_1,a_2,b_1,b_2,c_1,c_2\}$ spans this K_6, where $A=\{a_1,a_2\},B=\{b_1,b_2\}$ and $C=\{c_1,c_2\}$ and let $Y=V(K_{18})\setminus X$. Now we observe the following:

- $|X_R(y)| \geq 2$ for all vertices $y \in Y$.
- If $|X_R(y_1)| = 2 = |X_R(y_2)|$ for two vertices $y_1, y_2 \in Y$ then $X_R(y_1) = X_R(y_2)$ or $X_R(y_1) \cap X_R(y_2) = \emptyset$, since there would be a blue $K_7 - 2P_2$ in $\langle B \rangle$ otherwise.
- Assume that $|X_R(y_i)| = 2$ for three vertices $y_1, y_2, y_3 \in Y$ and $X_R(y_1) = X_R(y_2) = X_R(y_3)$, say $X_R(y_1) = \{a_1, a_2\}$. Then $\{y_1, y_2, y_3, b_1, b_2, c_1, c_2\}$ spans a blue K_7 and hence we conclude $|X_R(y)| = 2$ for at most six vertices in Y.
- There are $5 \times 6 = 30$ red edges between X and Y.

Let $Z = \{y \in Y \mid |X_R(y)| = 2\}$ and $W = Y \setminus Z$. Altogether it follows that the following configuration — where $|Z| = 6$ and $|X_R(w)| = 3$ for all $w \in W$ – is the only possible one:



Avoiding a blue $K_7 - P_2$ we also know that $|A_R(w)| = 1, |B_R(w)| = 1$ and $|C_R(w)| = 1$ for all $w \in W$. In particular this implies $wz \in \langle B \rangle$ for all $w \in W$, $z \in Z$. Assume that $X_R(w_1) = \{a_1, b_1, c_1\}$. Since $|N_R(w_1)| = 5$ and $w_1 z \in \langle B \rangle$ for all $z \in Z$ the vertex w_1 has two red neighbours in W, say $w_1 w_2$ and $w_1 w_3$ are red. Now the set $\{z_1, a_2, b_2, c_2, w_2, w_3\}$ spans a blue $K_6 - P_2$. The vertex z_1 has no red neighbour in W and exactly two in X. Hence there is a vertex in $\{z_3, z_4, z_5, z_6\}$ which is in blue adjacent to z_1, say z_3. But this implies that there is a blue $K_7 - 2P_2$ — spanned by $\{z_1, z_3, a_2, b_2, c_2, w_2, w_3\}$.

Altogether we get a contradiction to case (1). In the sequel we have at least two vertices $v \in V(K_{18})$ with exactly six red neighbours, say $|N_R(a)| = 6 = |N_R(b)|$.

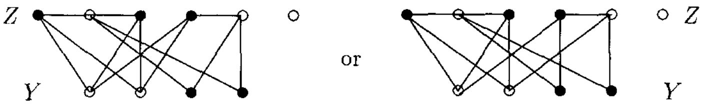
(2) $|N_R(a) \cap N_R(b)| = 1$: In this case $N_R(a) \cup \{b\}$ spans a blue $K_7 - P_2$.

(3) $|N_R(a) \cap N_R(b)| = 5$: This time $N_R(a) \cup N_R(b)$ spans a blue $K_7 - P_2$.

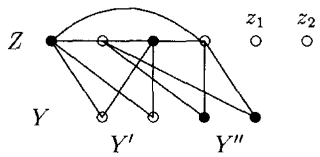
(4) $|N_R(a) \cap N_R(b)| = 6$: Let $X = N_R(a) = N_R(b)$ and $W = V(K_{18}) \setminus (X \cup \{a, b\})$. Now the following holds:

- Since $|X_R(w)| \geq 2$ for all $w \in W$ there are at least $2 \times 10 = 20$ edges between X and W and thus there is a vertex $x \in X$ with $|W_R(x)| = 4$. We assume $|W_R(x_6)| = 4$ ($|W_R(x)| \geq 5$ for any vertex $x \in X$ would give a blue K_7) and put $Y = W_R(x_6)$ and $Z = W \setminus Y$.
- $|Y_R(z)| \geq 2$ for all $z \in Z$, since $Y \cup \{a, b\}$ spans a K_6 in $\langle B \rangle$.
- Avoiding that $Z \cup \{a, b\}$ spans a blue $K_7 - 2P_2$ the set Z contains at least a red $P_3 \cup P_2$ or a red C_4 (remember that $az, bz \in \langle B \rangle$ for all $z \in Z$).

(a) $P_3 \cup P_2 \subseteq \langle R \rangle_Z$: Since $|Y_R(z)| \geq 2$ for all $z \in Z$ and, since there is no red triangle, there are at least the following red edges in $Z \cup Y$ and in each case the filled vertices together with $\{a,b\}$ span a K_7 or a $K_7 - P_2$ in $\langle B\rangle$:



(b) $C_4 \subseteq \langle R\rangle_Z$: This time there are at least the following red edges in $Z \cup Y$:



Avoiding that z_1 (z_2 resp.) together with $\{a,b\}$ and either the filled or the unfilled vertices spans a blue $K_7 - P_2$, we conclude that $Y_R(z_1) = Y = Y_R(z_2)$. Since $|X_R(z_1)| \geqslant 2$, say $\{x_1,x_2\} \subseteq X_R(z_1)$, there is a blue K_6 spanned by $N_R(z_1)$. Again avoiding a blue $K_7 - P_2$ we note that $|Y_R(x_i)| \geqslant 2$ for $i = 3,4,5$. This implies $|N_R(y)| \geqslant 7$ ($|Z_R(y)| = 4$, $|X_R(y)| \geqslant 3$) for at least one $y \in Y$ which gives a blue K_7.

(5) $|N_R(a) \cap N_R(b)| = 2$: Let $N_R(a) = \{x_1,x_2,x_3,x_4,z_1,z_2\}$, $N_R(b) = \{z_1,z_2,x_5,x_6,x_7,x_8\}$, $X^1 = \{x_1,x_2,x_3,x_4\}$, $X^2 = \{x_5,x_6,x_7,x_8\}$, $X = X^1 \cup X^2$ and $Y = V(K_{18}) \setminus (\{a,b\} \cup N_R(a) \cup N_R(b))$. Now the following holds:

- Both yz_1 and yz_2 are red edges or $|X_R^1(y)| \geqslant 2$ and $|X_R^2(y)| \geqslant 2$ for all $y \in Y$, since otherwise there would be a blue $K_7 - 2P_2$ spanned by $\{a\} \cup X^2 \cup \{y\} \cup \{z_i\}$ or by $\{b\} \cup X^1 \cup \{y\} \cup \{z_i\}$ for $i = 1$ or 2.
- Let $Y' = \{y \in Y \mid yz_1, yz_2 \in \langle R\rangle\}$. If $|Y'| \geqslant 4$ then we continue like in case (4), with $a = z_1$ and $b = z_2$. Hence $|Y'| \leqslant 3$ and there is a vertex $y_1 \in Y \setminus Y'$. Suppose without loss of generality that $y_1x_3, y_1x_4, y_1x_5, y_1x_6 \in \langle R\rangle$.
- If y_1 has another red neighbour in X, say x_2, then the set $\{x_2,x_3,x_4,x_5,x_6,z_1,z_2\}$ spans a blue K_7. Hence $|X_R^1(y)| = 2$ and $|X_R^2(y)| = 2$ for all $y \in Y \setminus Y'$.
- Consider a vertex $y_2 \notin Y'$. If $X_R(y_1) \cap X_R(y_2) = \emptyset$ then there is a blue $K_7 - P_2$ spanned by $\{x_3,x_4,x_5,x_6,z_1,z_2,y_2\}$, hence let $x_6 \in X_R(y_2)$. Avoiding that $\{x_1\}$ ($\{x_2\}$ resp.) spans a blue $K_7 - P_2$ together with $\{x_3,x_4,x_5,x_6,z_1,z_2\}$ we obtain $x_1,x_2 \notin X_R(y_2)$ and thus $x_3,x_4 \in X_R(y_2)$. Analogously it follows that $x_7,x_8 \notin X_R(y_2)$, but $x_5 \in X_R(y_2)$ which means $X_R(y_1) = X_R(y_2)$. Moreover, we conclude that $X_R(y) = X_R(y_1)$ for all $y \in Y \setminus Y'$.

If $|Y'| = 3$ (put $Y' = \{y_4,y_5,y_6\}$) then $\{a,b,y_4,y_5,y_6\} \subseteq N_R(z_1)$ and $\{a,b,y_4,y_5,y_6\} \subseteq N_R(z_2)$. Now z_1 as well as z_2 has at most one red neighbour in $\{y_1,y_2,y_3\}$. If these are different (or if any does not have such a neighbour) then $\{x_1,x_2,z_1,z_2,y_1,y_2,y_3\}$ spans a blue $K_7 - 2P_2$. If both z_1 and z_2 have the same common neighbour then $|Y'| \geqslant 4$ which contradicts our assumption.

If $|Y'| \leqslant 2$ then there are two more vertices $y_3,y_4 \notin Y'$ and it follows that $X_R(y_3) = X_R(y_4) = X_R(y_1)$. Now the set $\{a,b,y_1,y_2,y_3,y_4,x_1\}$ spans a blue $K_7 - P_2$.

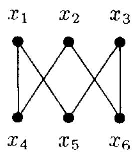
(6) $|N_R(a)\cap N_R(b)|=4$: Let $N_R(a)=\{x_1,x_2,z_1,z_2,z_3,z_4\}$, $N_R(b)=\{z_1,z_2,z_3,z_4,x_3,x_4\}$, $Z=\{z_1,z_2,z_3,z_4\}$ and $Y=V(K_{18})\setminus(N_R(a)\cup N_R(b)\cup\{a,b\})$. Now we observe the following:

- If $x_1y\in\langle R\rangle$ ($x_2y\in\langle R\rangle$ resp.) then $x_3,x_4\notin X_R(y)$ for all $y\in Y$, since otherwise $\{x_1,z_1,z_2,z_3,z_4,x_3,x_4\}$ would span a blue K_7-P_2. Therefore $|Z_R(y)|\geqslant 2$ for all $y\in Y$ and thus $|Y_R(z)|\geqslant 4$ (actually $|Y_R(z)|=4$ because there is no blue K_7).
- $1\leqslant|Y_R(z_i)\cap Y_R(z_j)|\leqslant 2$ for $i\neq j$, because otherwise we get a contradiction following one of the previous cases depending on the value of $|Y_R(z_i)\cap Y_R(z_j)|$.

(a) Suppose $|Y_R(z_i)\cap Y_R(z_j)|=2$ for at least one pair (i,j), say $Y_R(z_1)=\{y_1,y_2,y_3,y_4\}$ and $Y_R(z_2)=\{y_3,y_4,y_5,y_6\}$. To avoid that $\{a,b\}\cup Y_R(z_1)\cup\{y_i\}$ for $i=5,6$ or $\{a,b\}\cup Y_R(z_2)\cup\{y_i\}$ for $i=1,2$ span a K_7-P_2 in $\langle B\rangle$, the edges $y_5y_1,y_6y_1,y_5y_2,y_6y_2$ are red. Since there is no red triangle either we conclude $\{y_7,y_8\}\subseteq Y_R(z_3)\cap Y_R(z_4)$. If $Y_R(z_3)=\{y_1,y_2,y_7,y_8\}$ then $Y_R(z_2)\cap Y_R(z_3)=\emptyset$. Analogously we get a contradiction if $Y_R(z_3)=\{y_5,y_6,y_7,y_8\}$ or if we replace z_3 with z_4. Hence we suppose $Y_R(z_3)=\{y_2,y_3,y_7,y_8\}$ and $Y_R(z_4)=\{y_4,y_5,y_7,y_8\}$. Then $\{a,b,y_2,y_3,y_5,y_7,y_8\}$ spans a blue K_7-P_2.

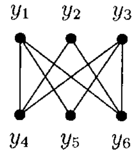
(b) $|Y_R(z_i)\cap Y_R(z_j)|=1$ for all $i\neq j$ contradicts $|Z|=4$ and $|Y|=8$.

(7) $|N_R(a)\cap N_R(b)|=3$: Let $N_R(a)=\{x_1,x_2,x_3,z_1,z_2,z_3\}$, $N_R(b)=\{z_1,z_2,z_3,x_4,x_5,x_6\}$, $X^1=\{x_1,x_2,x_3\}$, $X^2=\{x_4,x_5,x_6\}$, $X=X^1\cup X^2$, $Z=\{z_1,z_2,z_3\}$ and $Y=V(K_{18})\setminus(N_R(a)\cup N_R(b)\cup\{a,b\})$. Now we derive contradictions considering the red edges.



- Avoiding a blue K_7-2P_2 in $Z\cup X^1\cup\{x_i\}$ for $i=4,5,6$ and also in $Z\cup X^2\cup\{x_i\}$ for $i=1,2,3$ we conclude $|X_R(x)|\geqslant 2$ for all $x\in X$. Hence there are at least the following edges between X^1 and X^2:
- Avoiding that $\{x_1,x_3,x_4,x_6,z_1,z_2,z_3\}$ spans a blue K_7-2P_2 we suppose that x_1x_6 is a red edge. Also, since otherwise $\{x_2,x_3,x_4,x_5,z_1,z_2,z_3\}$ spans a blue K_7-2P_2, we suppose $x_3x_4\in\langle R\rangle$.
- Each $y\in Y$ has at least two red neighbours in $X^1\cup Z$ and in $X^2\cup Z$. But $yx_i\in\langle R\rangle$ for $i=1$ or 3 implies that $yx_i\notin\langle R\rangle$ for $i=4,5,6$ which gives that y has at least two red neighbours in Z. Analogously, from $yx_i\in\langle R\rangle$ for $i=4$ or 6 it follows that y has at least two red neighbours in Z.
  Altogether we conclude that yx_2 and $yx_5\in\langle R\rangle$ or that y has at least two red neighbours in Z.
- Since $|N_R(x_2)|\leqslant 6$ there are at least 11 red edges between Y and Z. Let $|Y_R(z_1)|=4=|Y_R(z_2)|$ (consider that $|N_R(v)|\leqslant 6$ for all $v\in V$) and according to (2)–(6) we have $|Y_R(z_1)\cap Y_R(z_2)|=1$, say $|Y_R(z_1)|=\{y_1,y_2,y_3,c\}$ and $|Y_R(z_2)|=\{c,y_4,y_5,y_6\}$. For symmetric reasons ($|N_R(z_1)|=6=|N_R(z_2)|$, $|N_R(z_1)\cap N_R(z_2)|=3$) we know that the following edges between $Y^1=\{y_1,y_2,y_3\}$ and $Y^2=\{y_4,y_5,y_6\}$

are red:



If one of the vertices y_1, y_3, y_4 and y_6 has only one red neighbour in Z – and hence has a red edge to x_2 and to x_5 — then $|N_R(y_i)|=6$ and $|N_R(y_i)\cap N_R(a)|=2$ for $i=1,3,4,6$. This contradicts case (5), thus $|Z_R(y_i)|\geqslant 2$ for all $i=1,3,4,6$. By the choice of Y^1 and Y^2 also $|Z_R(c)|\geqslant 2$ holds. Then there are at least 12 red edges between Y and Z which means that $|N_R(z_3)|\geqslant 6$ and $|N_R(z_i)\cap N_R(z_3)|\neq 3$ for $i=1$ or 2.

(8) $N_R(a)\cap N_R(b)=\emptyset$: If a and b are nonadjacent in $\langle R\rangle$ then $N_R(a)\cup\{b\}$ spans a blue K_7. Hence $ab\in\langle R\rangle$ and let $X^1=N_R(a)\setminus\{b\}=\{x_1,\dots,x_5\}$, $X^2=N_R(b)\setminus\{a\}=\{x_6,\dots,x_{10}\}$ and $Y=V(K_{18})\setminus(X^1\cup X^2\cup\{a,b\})$. From the cases (2)–(7) we conclude $|N_R(v)|=5$ for all $v\in V(K_{18})\setminus\{a,b\}$. Avoiding a blue K_7-2P_2 we also know $|X_R^1(y)|\geqslant 2$ and $|X_R^2(y)|\geqslant 2$ for all $y\in Y$ which means that each $y\in Y$ has at most one red neighbour in Y. Also avoiding that $Y\cup\{a,b\}$ contains a blue K_7-2P_2 we conclude that Y contains exactly three independent red edges, say $y_1y_2, y_3y_4, y_5y_6\in\langle R\rangle$. Thus $|X_R^1(y)|=2$ and $|X_R^2(y)|=2$ for all $y\in Y$ which implies that there are 12 red edges between X^1 and Y and 12 red edges between X^2 and Y. Also each vertex $x\in X^1$ has at least one red neighbour in X^2 and vice versa. Now there is one vertex in X^1, say x_5, which has exactly three red neighbours in Y and exactly one in X^2, so let x_5y_1, x_5y_3, x_5y_5 and x_5x_6 be red edges. The vertex x_6 has at least two red neighbours in Y, say y_4 and y_6 (remember that there is no red triangle). Since $|X_R^2(y)|=2$ for all $y\in Y$ we know that y_4 has exactly one additional red neighbour in X^2, thus the set $\{a,x_5,x_7,x_8,x_9,x_{10},y_4\}$ spans a blue K_7-2P_2 where the edges ax_5 and y_4x_i for some $i\in\{8,9,10\}$ are red. This gives our last contradiction and hence we conclude that $r(K_3,K_7-2P_2)\leqslant 18$. □

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